Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
International Journal of Solids and Structures 41 (2004) 6505–6528
INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES
www.elsevier.com/locate/ijsolstr

Transient resonant interactions of finite linear chains with essentially nonlinear end attachments leading to passive energy pumping

Panagiotis N. Panagopoulos, Alexander F. Vakakis *, Stylianos Tsakirtzis

School of Applied Mathematical and Physical Sciences, National Technical University of Athens, P.O. Box 64042, 15710 Zografosa, Athens, Greece

Received 3 August 2003; received in revised form 27 April 2004
Available online 10 June 2004

Abstract

We study resonance capture phenomena leading to energy pumping in systems with multiple degrees of freedom (DOF), composed of N linear oscillators weakly coupled to strongly nonlinear attachments possessing essential (nonlinearizable) cubic stiffness nonlinearities. First we present numerical evidence of energy pumping in the systems under consideration, i.e., of passive, one-way (irreversible) transfer of externally imparted energy to the nonlinear attachments, provided that the energy is above a critical level. To obtain a better understanding of the energy pumping phenomenon we reduce the dynamics governing the chain–attachment interaction to a single, nonlinear integro-differential equation that governs exactly the transient dynamics of the strongly nonlinear attachment. By introducing an approximation based on Jacobian elliptic functions we derive an approximate set of two nonlinear integro-differential modulation equations that govern the time evolution of the amplitude and phase of the motion of the attachment. This set of modulation equations is studied both analytically and numerically.

We then perform a perturbation analysis in an $O(\sqrt{\varepsilon})$ neighborhood of a 1:1 resonant manifold of the system in order to study the attracting region in the reduced phase space of the system, that is responsible for resonance capture and nonlinear energy pumping. This analysis provides a justification of the numerical findings.

1. Introduction

In previous works (Vakakis and Gendelman, 2001; Vakakis et al., 2003) *passive nonlinear energy pumping* of broadband vibration energy from a main (linear) damped structure to a damped, essentially nonlinear attachment has been studied. It was shown that above a critical energy threshold the nonlinear attachment

* Corresponding author. Address also at: Department of Mechanical and Industrial Engineering (adjunct), University of Illinois at Urbana, Champaign. Tel.: +30-21-07721361; fax: +30-21-07721302.

E-mail addresses: panago58@otenet.gr (P.N. Panagopoulos), avakakis@uiuc.edu, vakakis@central.ntua.gr (A.F. Vakakis), tsaksel@central.ntua.gr (S. Tsakirtzis).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2004.05.005

passively absorbs and confines energy from the linear structure, acting in essence as a *nonlinear energy sink.* Energy pumping is caused by *1:1 resonance capture* (Arnold, 1988; Quinn, 1997, 2002). Vakakis et al. (2003).

In this work resonance capture (and passive energy pumping) is studied for a system composed of a finite chain of particles with a weakly coupled, essentially nonlinear attachment. We reduce the dynamics to a single strongly nonlinear integro-differential equation governing the oscillation of the attachment, and study the dynamics by asymptotic techniques. These asymptotic techniques are non traditional since they are applied to analyze essentially nonlinear (nonlinearizable) transient (damped) dynamics; this dictated the development of new analytical methods, capable of modelling the strongly nonlinear regimes considered.

This work contributes towards the development of a new paradigm for passively controlling vibration and shock in engineering structures. In essence, this new paradigm is based on passively channelling unwanted energy of vibration into local nonlinear attachments also termed nonlinear energy sinks (NESs), where this energy is confined and dissipated. The proposed design differs in a number of aspects from existing ones. In contrast to classical linear vibration absorbers, which are narrowband devices (e.g., they are effective in the neighbourhood of a single frequency) the proposed nonlinear local attachments are capable of passively absorbing and dissipating broadband (transient) disturbances. Moreover, as shown in this work the proposed attachments can nonlinearly interact with a series of structural modes, extracting a significant amount of energy from each before engaging the next; this phenomenon (which is due to resonance capture cascades) is rather unique for the class of passive dynamic absorption devices considered herein. In essence, the proposed local attachments act as passive, adaptive, boundary controllers.

In contrast to existing works in this field, we consider and analytically study general transient, strongly nonlinear responses, and the techniques developed directly address the transient problem (and not the steady-state as in the majority of existing works). We note that the proposed designs have wide applicability to diverse problems encompassing many engineering disciplines, such as mechanical (vibration and shock isolation of machines, packaging), civil (seismic mitigation) and aerospace (disturbance isolation of sensitive devices during launch of payloads in space, flutter suppression). What contributes to the practicality of the proposed NES design is, modularity (they can be connected to existing structures with minimal structural modification), simplicity and passivity (does not require power to operate), and its relative inexpensiveness compared to traditional structural redesigns.

2. Formulation of the problem and numerical evidence

The system under consideration is a finite chain of N particles with linear grounding stiffnesses, undergoing linear next-neighbor interactions. The chain is coupled at its right boundary to a strongly nonlinear, weakly damped oscillator (attachment). We wish to study the nonlinear interaction of the chain with the attachment, and, in particular, nonlinear energy transfer exchanges resulting from this interaction. The set of equations governing the dynamics is as follows:

$$\begin{aligned}
&\ddot{u}_1 + u_1(\omega_0^2 + 2\alpha) - \alpha u_2 = 0\\
&\ddot{u}_n + u_n(\omega_0^2 + 2\alpha) - \alpha(u_{n+1} - u_{n-1}) = 0, \quad n = 2, 3, \ldots, N-1\\
&\ddot{u}_N + u_N(\omega_0^2 + \alpha + \varepsilon) - \alpha u_{N-1} - \varepsilon v = 0\\
&\ddot{v} + Cv^3 + \varepsilon\beta\dot{v} + \varepsilon(v - u_N) = 0
\end{aligned} \tag{1}$$

where u_n denotes the displacement of the nth particle of the chain, v the displacement of the nonlinear oscillator, α the coupling between particles of the chain, β the damping coefficient of the nonlinear oscillator, and ω_0^2 the stiffness of the on-site (grounding) quadratic potential. The perturbation parameter $0 \ll \varepsilon \ll 1$ scales the weak coupling between the chain and the nonlinear oscillator, and the parameter C

denotes the strength of the essential (nonlinearizable) stiffness nonlinearity. As usual, dot denotes differentiation with respect to time, and the particles are assumed to be of unit mass. For purpose of reference we will refer to the following system with grounded boundary condition instead of the nonlinear attachment as the 'linear chain':

$$\begin{aligned} &\ddot{u}_1 + u_1(\omega_0^2 + 2\alpha) - \alpha u_2 = 0 \\ &\ddot{u}_n + u_n(\omega_0^2 + 2\alpha) - \alpha(u_{n+1} - u_{n-1}) = 0, \quad n = 2, 3, \ldots, N-1 \\ &\ddot{u}_N + u_N(\omega_0^2 + \alpha + \varepsilon) - \alpha u_{N-1} = 0 \end{aligned} \tag{2}$$

To analyze the nonlinear interaction between the attachment and the chain in (1) it is instructive to initially compute the approximate instantaneous frequency of the nonlinear oscillator during the motion. This was analytically computed in (Vakakis and Gendelman, 2001) using the action-angle transformation of the uncoupled nonlinear attachment:

$$\Omega(t) = \Xi I^{1/3}(t)$$

$$\Xi = \left(\frac{3\pi^4 C}{8K(1/2)^4}\right)^{1/3}, \quad I(t) = \left(\frac{\pi^2 \dot{v}^2(t)}{2\Lambda^2 \Xi K(1/2)} + \frac{v^4(t)}{\Lambda^4}\right), \quad \Lambda = \left(\frac{1}{4C}\right)^{1/6}\left(\frac{3\pi}{K(1/2)}\right)^{1/3} \tag{3}$$

where $K(1/2)$ is the complete elliptic integral of the first kind with modulus 1/2.

A numerical computation based on a two-DOF chain ($N = 2$) demonstrates some important issues of the chain–attachment interaction. In Fig. 1 we depict the transient responses and the approximate instantaneous frequency $\Omega(t)$ of the nonlinear oscillator for a system with parameters $\alpha = 1$, $\omega_0 = 1$, $\beta = 2$,

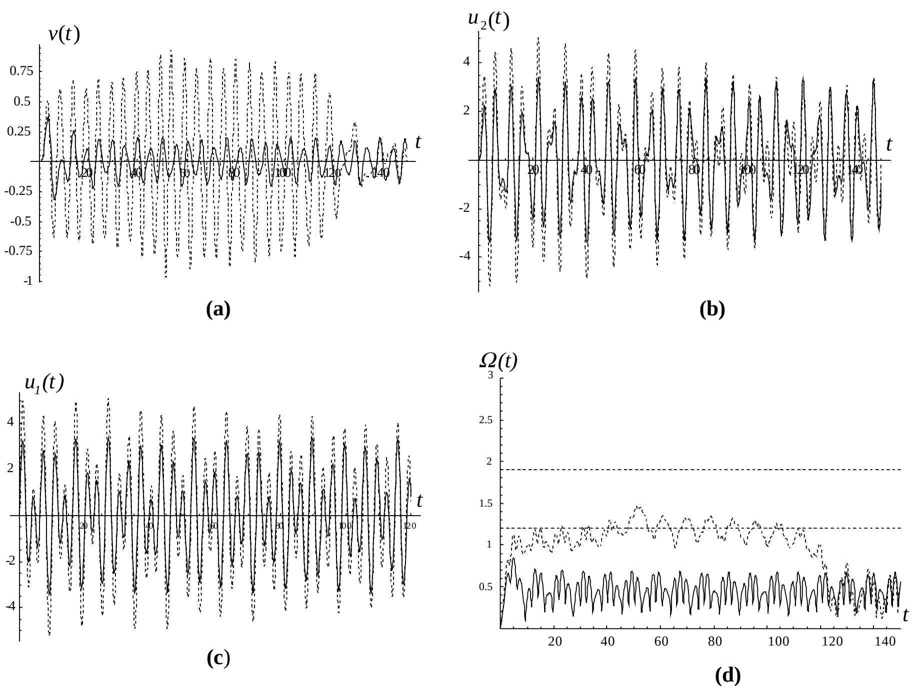

Fig. 1. Numerical transient responses of the two-DOF system with nonlinear attachment: (a) $v(t)$, (b) $u_2(t)$, (c) $u_1(t)$, (d) Instantaneous frequency $\Omega(t)$; (–) $Y = 5.6$; (- - -) $Y = 8.6$.

$C = 3$ and $\varepsilon = 0.1$. We used zero initial conditions except for $\dot{u}_1(0) = Y$; these initial conditions correspond to impulsive excitation at $t = 0$ of the farthest from the attachment oscillator 1. For $Y = 5.6$ no significant nonlinear interaction between the chain and the attachment takes place, and most of the induced energy of vibration remains in the linear part of the system, where it is originally generated. However, by increasing the initial condition to $Y = 8.6$ strong energy transfer to the nonlinear attachment is observed.

The enhanced nonlinear interaction as energy increases can be better understood by considering the plots of Fig. 1d, depicting the variation of the instantaneous frequency $\Omega(t)$ of the nonlinear attachment for the two aforementioned cases of impulsive excitation. Indeed, for the lower level excitation $\Omega(t)$ does not reach the neighbourhood of the natural frequencies of the linear chain, and, as a result no resonance interaction (capture) between the attachment and the chain can occur. By contrast, for the case of higher impulsive excitation the instantaneous frequency of the nonlinear oscillator reaches the neighborhood of the smallest natural frequency ω_1 of the linear chain giving rise to *1:1 resonance capture* (Vakakis and Gendelman, 2001). By this we mean the *transient* internal resonance between the attachment and the chain in a small neighborhood of a 1:1 resonant manifold of the dynamics (Arnold, 1988; Quinn, 1997, 2002). Hence, the nonlinear attachment engages in a 1:1 transient resonance interaction with the lowest mode of the linear chain, during which one-way transfer (pumping) of energy to the attachment takes place.

By increasing the magnitude of the impulse to $Y = 25$, there occurs a resonance capture cascade, whereby the attachment transiently resonates with both modes of the linear chain in sequential order. This can be concluded from the frequency plot of Fig. 2, where it is seen that the instantaneous frequency $\Omega(t)$ of the nonlinear oscillator first reaches the neighbourhood of the natural frequency of the higher, anti-phase mode of the linear chain, and then makes the transition to the neighbourhood of the natural frequency of the lower, in-phase mode. Damping dissipation is the mechanism that reduces continuously the overall energy of the system and induces the frequency transitions. After the resonance capture cascade the frequency fluctuates about a small mean value, and decays to zero as energy diminishes. The resonance capture cascade and the low energy behaviour of the instantaneous frequency will be analyzed in the next section.

3. Analytical treatment of transient resonance interactions

We now perform an analytic investigation of the transient resonant interactions of the nonlinear attachment with the modes of the linear chain. First, we consider the chain and consider the force exerted by the coupling stiffness to be a pseudo-forcing term. The equations of motion for the N particles of chain are given by:

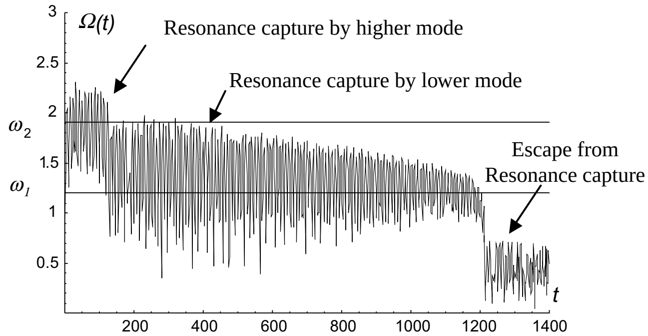

Fig. 2. Instantaneous frequency $\Omega(t)$ of the two-DOF system for $Y = 25$.

$$\begin{aligned}&\ddot{u}_1 + u_1(\omega_0^2 + 2\alpha) - \alpha u_2 = 0\\&\ddot{u}_n + u_n(\omega_0^2 + 2\alpha) - \alpha(u_{n+1} - u_{n-1}) = 0, \quad n = 2, 3, \ldots, N-1\\&\ddot{u}_N + u_N(\omega_0^2 + \alpha + \varepsilon) - \alpha u_{N-1} = \varepsilon v\end{aligned} \tag{4}$$

Later we will consider separately the nonlinear differential equation governing the motion of the nonlinear attachment. Due to its linear structure, system (4) can be placed in the following matrix form:

$$I\ddot{u} + Ku = f(t), \quad u(t) \in \Re^N \tag{5}$$

with initial conditions $u(0)$ and $\dot{u}(0)$. Variables in capitals denote $N \times N$ matrices, whereas variables in lowercase denote $N \times 1$ vectors; I is the $N \times N$ unit matrix, and the $N \times 1$ pseudo-force vector is given by $f(t) = [000 \cdots 0 \ \varepsilon v(t)]^{\mathrm{T}}$.

To solve system (5), we express the displacement vector in the series form,

$$u(t) = \sum_{i=1}^{N} q_i(t)\varphi_i \tag{6}$$

where $q_i(t)$ are modal amplitudes, and φ_i, $i = 1, \ldots, N$, are is the mass-normalized, mutually orthogonal eigenvectors of the stiffness matrix K of the linear chain with associated eigenvalues ω_i^2. The eigensolutions satisfy the well known conditions,

$$\varphi_i^{\mathrm{T}}\varphi_j = \delta_{ij}, \quad \varphi_i^{\mathrm{T}}K\varphi_j = \omega_i^2\delta_{ij} \tag{7}$$

where δ_{ij} is the Kronecker delta. Substituting (6) into (5), pre-multiplying by φ_i^{T}, and utilizing (7) we obtain the following N decoupled equations governing the unknown functions $q_i(t)$:

$$\ddot{q}_i + \omega_i^2 q_i = \varphi_i^{\mathrm{T}} f(t) = \varepsilon\varphi_{i,N} v(t), \quad i = 1, \ldots, N \tag{8}$$

where $\varphi_{i,N}$ denotes the N-component of the ith eigenvector φ_i. The solution to each decoupled equation of the set (8) is expressed as,

$$q_i(t) = q_i(0)\cos(\omega_i t) + \frac{\dot{q}_i(0)}{\omega_i}\sin(\omega_i t) + \varepsilon\frac{\varphi_{i,N}}{\omega_i}\int_0^t v(\lambda)\sin\omega_i(t-\lambda)\,\mathrm{d}\lambda, \quad i = 1, \ldots, N \tag{9}$$

The initial conditions $q_i(0)$, $\dot{q}_i(0)$ are determined by,

$$q_i(0) = \varphi_i^{\mathrm{T}}u(0) = \sum_{k=1}^{N}\varphi_{i,k}u_k(0), \quad \dot{q}_i(0) = \varphi_i^{\mathrm{T}}\dot{u}(0) = \sum_{k=1}^{N}\varphi_{i,k}\dot{u}_k(0), \quad i = 1, \ldots, N \tag{10}$$

where $\varphi_{i,k}$ denotes the k-component of ith eigenvector φ_i and $u_k(0)$, $\dot{u}_k(0)$ denote the physical initial conditions.

Combining these results we express the displacement of lth particle of the chain by the following series expression, where the response of the nonlinear attachment appears as a pseudo-forcing term in the integral on the right hand side:

$$\begin{aligned}u_l(t) = {}&\sum_{i=1}^{N}\sum_{k=1}^{N}\varphi_{i,l}\varphi_{i,k}u_k(0)\cos(\omega_i t) + \sum_{i=1}^{N}\sum_{k=1}^{N}\frac{\varphi_{i,l}\varphi_{i,k}\dot{u}_k(0)}{\omega_i}\sin(\omega_i t)\\&+ \varepsilon\sum_{i=1}^{N}\frac{\varphi_{i,l}\varphi_{i,N}}{\omega_i}\int_0^t v(\lambda)\sin\omega_i(t-\lambda)\,\mathrm{d}\lambda, \quad l = 1, \ldots, N\end{aligned} \tag{11}$$

As a final step, we express $u_N(t)$ through (11) and substitute it into the last of the set of Eqs. (1). We then obtain a *single* essentially nonlinear, damped integro-differential equation that governs *exactly* the transient dynamics of the nonlinear attachment:

$$\ddot{v} + Cv^3 + \varepsilon\beta\dot{v} + \varepsilon v = \varepsilon\sum_{i=1}^{N}\sum_{k=1}^{N}\phi_{i,N}\phi_{i,k}u_k(0)\cos(\omega_i t) + \varepsilon\sum_{i=1}^{N}\sum_{k=1}^{N}\frac{\phi_{i,N}\phi_{i,k}\dot{u}_k(0)}{\omega_i}\sin(\omega_i t) + \varepsilon^2\sum_{i=1}^{N}\frac{\phi_{i,N}\phi_{i,N}}{\omega_i}\int_0^t v(\lambda)\sin\omega_i(t-\lambda)\,\mathrm{d}\lambda \tag{12a}$$

This equation is solved with initial conditions $v(0)$ and $\dot{v}(0)$. Hence, we have reduced the problem of nonlinear interaction of the chain with the nonlinear attachment to a single integro-differential equation. The first two terms on the right hand side represent the influence of the initial condition of the chain on the motion of the attachment; these are mere nonhomogeneous terms that do not pose any challenges from an analysis point of view. The summation term containing the integrals, however, models the interaction of the nonlinear attachment with the linear chain, including energy transfer into the attachment, and scattering of incoming waves from the nonlinear attachment, resulting in radiation of energy from the attachment back to the chain. To better understand the meaning of the integrals in that term, we performed a series of numerical simulations with (12) considering all initial energy in the nonlinear attachment [this amounted to setting $u_k(0) = \dot{u}_k(0) = 0$, $k = 1,\dots,N$ and considering nonzero initial conditions for the attachment]. For this type of initial conditions the integrals model radiation of energy from the attachment to the chain.

As an example, in Fig. 3 we depict the response of instantaneous frequency $\Omega(t)$ of the nonlinear oscillator, together with the integrals,

$$Y_{ci} = \int_0^t v(\lambda)\cos(\omega_i\lambda)\,\mathrm{d}\lambda \quad \text{and} \quad Y_{si} = \int_0^t v(\lambda)\sin(\omega_i\lambda)\,\mathrm{d}\lambda, \quad i = 1,2 \tag{12b}$$

for a two-DOF linear chain with a nonlinear attachment at its end; the parameters were assigned the values $\alpha = 1$, $\varepsilon = 0.01$, $\omega_0 = 1$, $C = 3$, $\beta = 1$, and initial conditions $u_1(0) = 0$, $u_2(0) = 0$, $v(0) = 1.5$, $\dot{u}_1(0) = 0$, $\dot{u}_2(0) = 0$ and $\dot{v}(0) = 0$. In this system a resonance capture cascade occurs, and the nonlinear attachment engages in resonance capture with both linearized modes in sequential order. We note that as instantaneous frequency $\Omega(t)$ approaches the neighborhood of each linearized natural frequency of the chain, the mean value of integrals corresponding to that natural frequency show large variations. This indicates that in the initial phase of the motion when the energy of the system is relatively high, the radiation of energy from the attachment to the linear chain excites the linearized modes in sequential order from high to low. As the energy of the nonlinear oscillator decreases due to radiation, so does its instantaneous frequency of oscillation, and the integrals cease to vary, reaching steady values. Hence, the nonlinear interaction of the attachment with the linearized modes of the chain can be traced in the behaviors of the integrals that characterize energy radiation from the attachment to the chain.

To analyze resonance capture and energy pumping from the N-DOF chain to the attachment, we transform the nonlinear integro-differential equation (12a) into a set of two first order nonlinear integro-differential equations that govern the time evolution of the amplitude and phase of the motion of the attachment. For this purpose we consider (12a) to be in the form of a nonhomogeneous nonlinear differential equation (13) in order to apply the method of variation of parameters to express its solution,

$$\ddot{v} + Cv^3 = \varepsilon f(t;\varepsilon) \tag{13a}$$

where,

$$f(t;\varepsilon) \equiv -v - \beta\dot{v} + \sum_{i=1}^{N}\sum_{k=1}^{N}\phi_{i,N}\phi_{i,k}u_k(0)\cos(\omega_i t) + \sum_{i=1}^{N}\sum_{k=1}^{N}\frac{\phi_{i,N}\phi_{i,k}\dot{u}_k(0)}{\omega_i}\sin(\omega_i t) + \varepsilon\sum_{i=1}^{N}\frac{\phi_{i,N}\phi_{i,N}}{\omega_i}\int_0^t v(\lambda)\sin\omega_i(t-\lambda)\,\mathrm{d}\lambda \tag{13b}$$

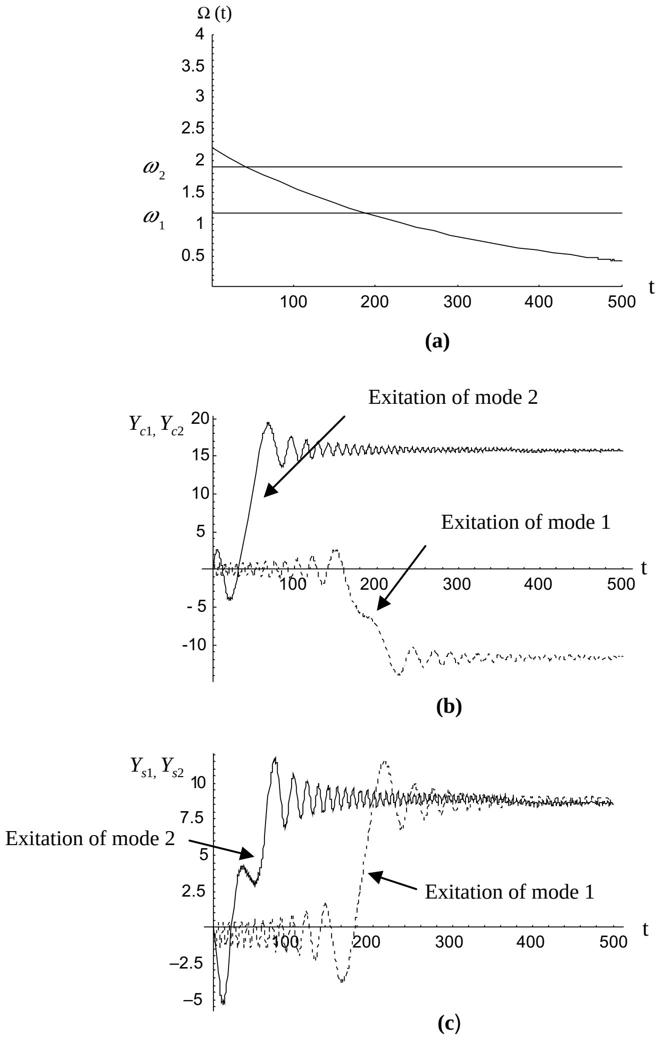

Fig. 3. Radiation of energy from the nonlinear attachment to a 2-DOF linear system with $\omega_1 < \omega_2$: (a) frequency $\Omega(t)$ of the attachment; (b) integrals Y_{c2} (–); Y_{c1} (- - -); (c) integrals Y_{s2} (–); Y_{s1} (- - -).

We first consider the solution of the homogeneous problem,

$$\ddot{v} + Cv^3 = 0 \Rightarrow v(t) = Acn[A\sqrt{C}t + 4K(1/\sqrt{2})\xi; 1/\sqrt{2}] \tag{14}$$

where A and ξ denote arbitrary constants, and $K(1/\sqrt{2})$ is the elliptic integral of the first kind.

Applying an 'elliptic' version of the method of variation of parameters (Barkham and Soudack, 1969, 1970; Coppola and Rand, 1990; Vakakis and Rand, 2004), we seek the solution of the inhomogeneous differential equation (13a) in the form,

$$v(t) = A(t)cn(A(t)\sqrt{C}t + 4K\xi(t); k) \tag{15}$$

where $A(t)$ and $\xi(t)$ are new unknown, *slowly varying amplitude and phase functions*, respectively. For simplicity the dependence of the variables v, A and ξ on the small variable ε is not explicitly depicted in Eq. (15).

In writing (15) we partition, in essence, the dynamics into fast and slow varying components, at the original time-scale t and at a time scale $\varepsilon^{\alpha}t$ ($a > 0$ to be determined), respectively. The plan is to average out the fast dynamics, and to confine the analysis to the slowly varying dynamics (the *slow flow*) where information on the attachment-chain nonlinear interaction is contained. The slowly varying amplitude and phase functions are governed by a set of slow flow modulation equations, which according to the method of variation of parameters are expressed as follows:

$$\begin{aligned}&\frac{\mathrm{d}A}{\mathrm{d}t}cn(u,k)+\frac{\mathrm{d}\xi}{\mathrm{d}t}4KA(t)cn'(u,k)=0\\&\frac{\mathrm{d}A}{\mathrm{d}t}2A(t)cn'(u,k)-\frac{\mathrm{d}\xi}{\mathrm{d}t}4K(A(t))^2(cn(u,k))^3=\frac{\varepsilon f(t;\varepsilon)}{\sqrt{C}}\end{aligned}\tag{16}$$

where,

$$cn'(u,k)\equiv\frac{\partial cn(u,k)}{\partial u},\quad \text{and}\quad u(t)\equiv A(t)\sqrt{C}t+4K\xi(t)$$

Solving this system in terms of the derivatives of the amplitude and phase functions, we obtain the following set of modulation equations:

$$\begin{aligned}&\frac{\mathrm{d}A}{\mathrm{d}t}=\varepsilon\frac{cn'}{A\sqrt{C}}f(t;\varepsilon)\\&\frac{\mathrm{d}\xi}{\mathrm{d}t}=-\varepsilon\frac{cn}{4KA^2\sqrt{C}}f(t;\varepsilon)\end{aligned}\tag{17}$$

We note that both derivatives are of $O(\varepsilon)$, indicating that indeed the amplitude and the phase are slowly varying functions.

Substituting (13b) into (17) we obtain the explicit form of the modulation equations of the slow flow:

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t}=\varepsilon\frac{cn'}{A\sqrt{C}}\Bigg\{&-A(t)cn(A\sqrt{C}t+4K\xi(t);k)-\beta\frac{\mathrm{d}}{\mathrm{d}t}\Big[A(t)cn(A\sqrt{C}t+4K\xi(t);k)\Big]\\&+\sum_{i=1}^{N}\sum_{k=1}^{N}\phi_{i,N}\phi_{i,k}u_k(0)\cos(\omega_i t)+\sum_{i=1}^{N}\sum_{k=1}^{N}\frac{\phi_{i,N}\phi_{i,k}\dot{u}_k(0)}{\omega_i}\sin(\omega_i t)\\&+\varepsilon\sum_{i=1}^{N}\frac{\phi_{i,N}\phi_{i,N}}{\omega_i}\int_0^t A(\lambda)cn(A\sqrt{C}\lambda+4K\xi(\lambda);k)\sin\omega_i(t-\lambda)\,\mathrm{d}\lambda\Bigg\}\end{aligned}\tag{18a}$$

$$\begin{aligned}\frac{\mathrm{d}\xi}{\mathrm{d}t}=-\varepsilon\frac{cn}{4KA^2\sqrt{C}}\Bigg\{&-A(t)cn(A\sqrt{C}t+4K\xi(t);k)-\beta\frac{\mathrm{d}}{\mathrm{d}t}\Big[A(t)cn(A\sqrt{C}t+4K\xi(t);k)\Big]\\&+\sum_{i=1}^{N}\sum_{k=1}^{N}\phi_{i,N}\phi_{i,k}u_k(0)\cos(\omega_i t)+\sum_{i=1}^{N}\sum_{k=1}^{N}\frac{\phi_{i,N}\phi_{i,k}\dot{u}_k(0)}{\omega_i}\sin(\omega_i t)\\&+\varepsilon\sum_{i=1}^{N}\frac{\phi_{i,N}\phi_{i,N}}{\omega_i}\int_0^t A(\lambda)cn(A\sqrt{C}\lambda+4K\xi(\lambda);k)\sin\omega_i(t-\lambda)\,\mathrm{d}\lambda\Bigg\}\end{aligned}\tag{18b}$$

Assuming that the *exact* response of nonlinear attachment is taken into account, in the form $v(t)=A(t)cn(A(t)\sqrt{C}t+4K\xi(t);1/\sqrt{2})$, Eqs. (18) provide the *exact* time evolution of the amplitude and phase of the motion of the nonlinear oscillator. However, since the exact analysis of the modulation Eqs. (18) is not possible, we resort to approximations in order to analytically study the nonlinear interaction of the attachment and the chain.

We simplify the problem by using the following Fourier expansion for the elliptic cosine,

$$cn(2K\theta/\pi) = 0.955\cos\theta + 0.043\cos 3\theta + \cdots \cong \cos\theta \tag{19}$$

and approximating the response of the nonlinear attachment by,

$$v(t) = A(t)\cos\theta(t) \tag{20}$$

where the angle variable is defined as,

$$\theta(t) = \frac{\pi\sqrt{C}}{2K}A(t)t + 2\pi\xi(t)$$

Expressing the modulation Eqs. (17) using the approximation (19) and the new angle variable, we obtain the following approximate modulation equations governing the slow flow of the system:

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= -\varepsilon\frac{\pi\sin\theta(t)}{2KA(t)\sqrt{C}}f(t;\varepsilon)\\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{\pi\sqrt{C}A(t)}{2K} - \varepsilon\frac{\pi\cos\theta(t)}{2K(A(t)^2)\sqrt{C}}f(t;\varepsilon)\end{aligned} \tag{21}$$

Taking into account the expression (13b) for $f(t)$, and approximating the velocity of the nonlinear attachment by the expressions,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t}\cos\theta - A\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{\pi\sqrt{C}A^2(t)}{2K}\sin\theta \tag{22}$$

we obtain the following approximate set of modulation equations governing the slow flow of the chain–nonlinear attachment interaction:

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= \varepsilon\frac{\pi}{2K\sqrt{C}}\cos\theta(t)\sin\theta(t) - \varepsilon\frac{\beta\pi^2A(t)}{4K^2}\sin^2\theta(t)\\ &\quad - \varepsilon\frac{\pi}{2KA(t)\sqrt{C}}\sin\theta(t)\sum_{i=1}^{N}\sum_{k=1}^{N}\phi_{i,N}\phi_{i,k}u_k(0)\cos(\omega_i t)\\ &\quad - \varepsilon\frac{\pi}{2KA(t)\sqrt{C}}\sin\theta(t)\sum_{i=1}^{N}\sum_{k=1}^{N}\frac{\phi_{i,N}\phi_{i,k}\dot{u}_k(0)}{\omega_i}\sin(\omega_i t)\\ &\quad - \varepsilon^2\frac{\pi}{2KA(t)\sqrt{C}}\sin\theta(t)\sum_{i=1}^{N}\frac{\phi_{i,N}\phi_{i,N}}{\omega_i}\int_0^t A(\lambda)\cos\theta(\lambda)\sin\omega_i(t-\lambda)\,\mathrm{d}\lambda + O(\varepsilon^3)\end{aligned} \tag{23a}$$

and,

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{\pi A(t)\sqrt{C}}{2K} + \varepsilon\frac{\pi}{2KA(t)\sqrt{C}}\cos^2\theta(t) - \varepsilon\frac{\beta\pi^2}{4K^2}\cos\theta(t)\sin\theta(t)\\ &\quad - \varepsilon\frac{\pi}{2K(A(t))^2\sqrt{C}}\cos\theta(t)\sum_{i=1}^{N}\sum_{k=1}^{N}\phi_{i,N}\phi_{i,k}u_k(0)\cos(\omega_i t)\\ &\quad - \varepsilon\frac{\pi}{2K(A(t))^2\sqrt{C}}\cos\theta(t)\sum_{i=1}^{N}\sum_{k=1}^{N}\frac{\phi_{i,N}\phi_{i,k}\dot{u}_k(0)}{\omega_i}\sin(\omega_i t)\\ &\quad - \varepsilon^2\frac{\pi}{2K(A(t))^2\sqrt{C}}\cos\theta(t)\sum_{i=1}^{N}\frac{\phi_{i,N}\phi_{i,N}}{\omega_i}\int_0^t A(\lambda)\cos\theta(\lambda)\sin\omega_i(t-\lambda)\,\mathrm{d}\lambda + O(\varepsilon^3)\end{aligned} \tag{23b}$$

From (23a) and (23b) it directly follows that $\frac{dA}{dt}$ is of order $O(\varepsilon)$ and $\frac{d\theta}{dt}$ is of order $O(1)$. The modulation equations approximately govern the amplitude and phase of the nonlinear attachment as it interacts with the modes of the linear chain. We stress that since we omitted terms of $O(\varepsilon^3)$ from (23), it is expected that their (transient) solutions will be valid only up to times of $O(1/\varepsilon^2)$. We first establish the accuracy of the approximate modulation equations by computing the response of the attachment by (20), and comparing it to direct numerical simulations of the original equations of motion (1). For this comparison we employ a two-DOF linear chain ($N = 2$) with a nonlinear attachment at its end, and parameters $\alpha = 1$, $\omega_0 = 1$, $\beta = 2$, $C = 3$, $\varepsilon = 0.1$; all initial conditions were taken as zero, with the exception $\dot{u}_1(0) = u$. In Fig. 4 we depict the comparison between the approximation (20) and direct numerical simulation for the transient response of the attachment $v(t)$ during or in the absence of energy pumping. Satisfactory agreement between the two responses is noted at least up to times of $O(1/0.1^2) = O(100)$; in fact for $t > 100$ a clear divergence between the exact and approximate results is observed in Fig. 4a, attributed to the fact that the approximate modulation equations (23) are correct only up to $O(\varepsilon^2)$.

In Fig. 5 the approximate amplitude $A(t)$ and phase $y(t) = \theta(t) - \omega_1 t$ (where ω_1 is the lower natural frequency of the linear chain) are depicted for a system with initial conditions zero except $\dot{u}_1(0) = u$. Direct numerical simulations established that for $u = 8.6$ resonance capture occurs in this system (see Fig. 1d). The same conclusion is reached by studying the approximate results of Fig. 5, where for $u = 8.6$ resonance capture with the first linearized mode of the chain close to the lowest linearized natural frequency ω_1 is established, leading to nonlinear energy pumping from the chain to the attachment; this is evidenced by the increased magnitude of the approximate amplitude $A(t)$ in the initial phase of the motion. Of interest is to

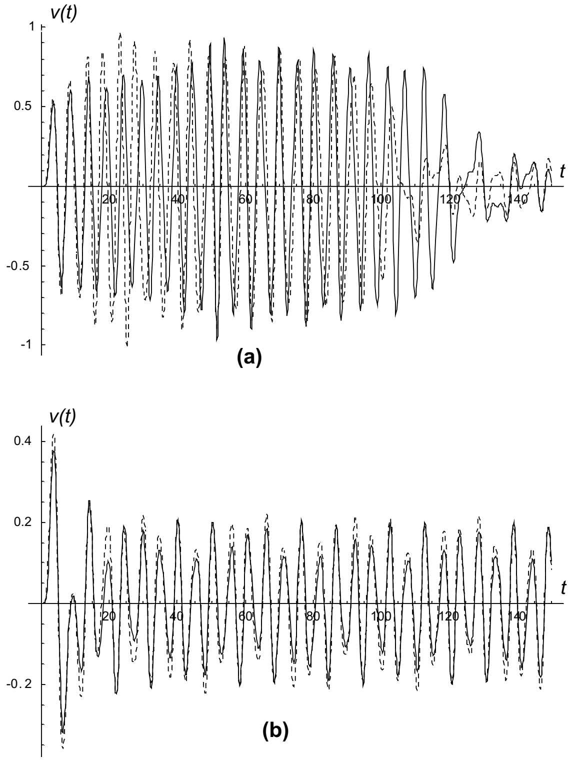

Fig. 4. Transient response of the nonlinear attachment: (a) when energy pumping takes place ($u = 8.6$), and (b) when no energy pumping takes place ($u = 5.6$): (---) analytic approximations based on modulation Eqs. (23a) and (23b), (–) direct numerical simulations.

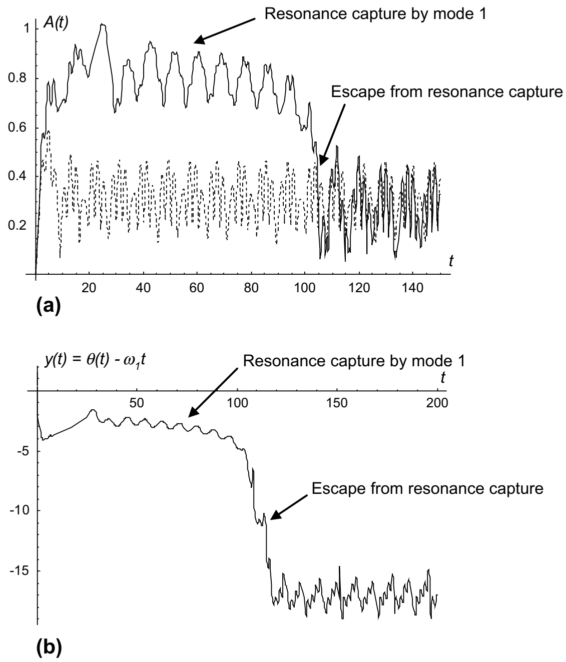

Fig. 5. Approximate transient response of the nonlinear attachment: (a) amplitude $A(t)$ for initial conditions $u = 5.6$ (- - -), and $u = 8.6$ (–); (b) phase difference for initial condition $u = 8.6$.

examine the transient evolution of the phase difference $y(t)$ for the case when resonance capture (and energy pumping) occurs ($u = 8.6$, Fig. 5b). We divide the evolution of $y(t)$ into three phases. In the first phase when resonance capture occurs there is a small-amplitude fast oscillation of $y(t)$ about a slowly-varying mean, indicating that when resonance capture occurs in the neighborhood of a natural frequency, there occurs a slow variation of the phase of the nonlinear attachment in the neighborhood of the linearized natural frequency of the resonant mode. In the second phase a rapid transition of $y(t)$ occurs away from the resonant natural frequency, and in third phase $y(t)$ is observed to perform small-amplitude oscillations about a small mean value. A perturbation analysis carried out in the Appendix A proves that in the third phase of the response the nonlinear attachment oscillates with a frequency equal to $\sqrt{\varepsilon(1 - \varepsilon\beta^2/4)}$ and effective damping ratio equal to $\zeta = \beta\varepsilon/2$. This result is consistent with the numerical instantaneous frequency simulation depicted in Fig. 2, where for sufficiently large times the frequency of the nonlinear attachment oscillates about a small mean value of $O(\sqrt{\varepsilon}) = O(0.316)$; the inability of the frequency plot of Fig. 5 to capture the correct frequency trend for large times is attributed to the previously discussed lack of validity of the approximate modulation equations (23) for large times.

Eqs. (23a) and (23b) together with Eq. (20) can be used to get insight into the dynamics of resonance capture and energy pumping. In particular, we wish to study the behavior of the integrals on the right-hand sides of expressions (23a) and (23b), which, as mentioned previously, model the interaction of the nonlinear attachment with the chain, including radiation of energy from the attachment back to the chain. All these integrals can be expressed in terms of the integrals Y_{ci} and Y_{si} defined by expressions (12b). Focusing in the region where resonance capture by mode 1 occurs (a similar analysis can be carried out for resonance capture by mode 2), it approximately holds that $\dot{\theta} \approx \omega_1$. It follows from (20) that in that region the displacement of the nonlinear attachment can be approximated by,

$$v(t) \approx A_1(t)\cos(\omega_1 t + c_1(t)) \text{ (resonance capture by mode 1)} \tag{24}$$

where $A_1(t)$ and $c_1(t)$ represent slowly varying amplitude and phase, which over the resonance capture region can be treated as being nearly constant. Substituting this approximation into the expression for the integral Y_{ci} (similarly we treat Y_{si}) we obtain the following approximation:

$$Y_{ci} \approx \int_0^t A_1(\lambda)\cos(\omega_1\lambda + c_1(t))\cos(\omega_i\lambda)\,\mathrm{d}\lambda \tag{25}$$

Considering $A_1(t)$ and $c_1(t)$ to be nearly constant, and using trigonometric identities, we may approximately express (25) as follows:

$$Y_{ci} \approx G_1(\lambda; c_1)\int_0^t \cos(\omega_1\lambda)\cos(\omega_i\lambda)\,\mathrm{d}\lambda + G_2(\lambda; c_1)\int_0^t \sin(\omega_1\lambda)\cos(\omega_i\lambda)\,\mathrm{d}\lambda \tag{26}$$

where $G_i(\lambda; c_1)$ are slowly varying functions of time. Using trigonometric identities we write (26) in the form:

$$Y_{ci} \approx G_1(\lambda; c_1)\left[\frac{\sin(\omega_1 - \omega_i)t}{2(\omega_1 - \omega_i)} + \frac{\sin(\omega_1 + \omega_i)t}{2(\omega_1 + \omega_i)}\right] + G_2(\lambda; c_1)\left[\frac{1 - \cos(\omega_1 - \omega_i)t}{2(\omega_1 - \omega_i)} - \frac{1 - \cos(\omega_1 + \omega_i)t}{2(\omega_1 + \omega_i)}\right] \tag{27}$$

It follows that if $i \neq 1$ the integral Y_{ci} performs low amplitude oscillations, whereas if $i = 1$ Y_{ci} undergoes large time rate variations. Similar conclusions hold for the integral Y_{si}. These results indicate that in the resonance capture region involving a certain mode of the linear chain, the attachment interacts strongly with that mode mainly through the scattering integrals in (23a) and (23b) which attain large rates of time variation; it is through such resonance interactions that energy pumping from the chain to the attachment takes place (Vakakis and Rand, 2004).

It is, therefore, of interest to study in more detail the set of modulation equations (23a) and (23b) in the region of resonance capture by imposing certain restrictions and assumptions on the slowly varying variables, thus focusing in the nonlinear interaction of the attachment with the chain. This is performed in the next Section where resort to local asymptotic analysis in the neighbourhood of the resonance capture region is carried out. The analysis follows resembles the asymptotic technique used in (Vakakis and Gendelman, 2001) where resonance capture interaction was studied by resorting to action-angle transformations. In the present work, however, the analysis is applied on the modulation equations of the slow flow, and results are sought in higher orders of approximation.

4. Analysis of resonance capture in the neighborhood of a resonance manifold

Our previews findings confirm that resonance capture (and energy pumping from the chain to the nonlinear attachment) in the dynamical system (1) occurs in the neighborhood of a 1:1 resonance manifold of the system, e.g., when the instantaneous frequency of the nonlinear attachment is close to one of the eigenfrequencies of the linear chain. Motivated by the above observations, we focus on the specific example of a two-DOF system composed of one linear oscillator weakly coupled to a weakly damped, strongly nonlinear oscillator, with parameters $\alpha = 1$, $\omega_0 = 1$, $\beta = 2$, $C = 3$, $\varepsilon = 0.1$ and zero initial values except for the velocity of the linear oscillator $\dot{u}(0) = u$:

$$\begin{aligned} &\ddot{u} + u(\omega_0^2 + \alpha + \varepsilon) - \varepsilon v = 0 \\ &\ddot{v} + Cv^3 + \varepsilon\beta\dot{v} + \varepsilon(v - u) = 0 \end{aligned} \tag{28}$$

We note that this by no means restricts the generality of the analysis, since resonance capture of the attachment with an arbitrary mode of an N-DOF linear chain can be reduced to the two-DOF interaction considered herein (Vakakis et al., 2003).

We will perform a *local analysis* by restricting the dynamics in an $O(\sqrt{\varepsilon})$ neighborhood of a 1:1 resonance manifold (Vakakis and Gendelman, 2001). To study the dynamics in the boundary layer close to this manifold we introduce the combination phase variable

$$\psi(t) = \theta(t) - \omega t \tag{29}$$

where $\omega = \sqrt{\alpha + \varepsilon + \omega_0^2}$, and the following amplitude transformation:

$$A(t) = R + \sqrt{\varepsilon} a(t) \tag{30}$$

where $R = 2K\omega(\pi\sqrt{C})^{-1}$. Substituting (29) and (30) into (23a) and (23b) with $N = 1$, using trigonometric identities and performing algebraic manipulations we obtain the following differential system of equations with respect to $a(t)$ and $\psi(t)$:

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t} =& \sqrt{\varepsilon}\frac{\pi}{4K\sqrt{C}}\sin 2(\psi + \omega t) - \sqrt{\varepsilon}\frac{\pi^2\beta[R + \sqrt{\varepsilon}a(t)]}{8K^2} + \sqrt{\varepsilon}\frac{\pi^2\beta[R + \sqrt{\varepsilon}a(t)]}{8K^2}\cos 2(\psi + \omega t) \\ & - \sqrt{\varepsilon}\frac{\pi u}{4K\sqrt{C}\omega[R + \sqrt{\varepsilon}a(t)]}\cos\psi + \sqrt{\varepsilon}\frac{\pi u}{4K\sqrt{C}\omega[R + \sqrt{\varepsilon}a(t)]}\sin(\psi + 2\omega t) \\ & - \varepsilon^{3/2}\frac{\pi\sin(\psi + \omega t)}{2K\sqrt{C}\omega[R + \sqrt{\varepsilon}a(t)]}\int_0^t [R + \sqrt{\varepsilon}a(\lambda)]\cos(\psi(\lambda) + \omega\lambda)\sin\omega_i(t - \lambda)\,\mathrm{d}\lambda\end{aligned} \tag{31a}$$

and,

$$\begin{aligned}\frac{\mathrm{d}\psi}{\mathrm{d}t} =& \sqrt{\varepsilon}\frac{\pi a(t)\sqrt{C}}{2K} + \varepsilon\frac{\pi}{4K[R + \sqrt{\varepsilon}a(t)]\sqrt{C}} + \varepsilon\frac{\pi}{4K[R + \sqrt{\varepsilon}a(t)]\sqrt{C}}\cos 2(\psi + \omega t) - \varepsilon\frac{\pi^2\beta}{8K^2}\sin 2(\psi + \omega t) \\ & + \varepsilon\frac{\pi u}{4K[R + \sqrt{\varepsilon}a(t)]^2\sqrt{C}\omega}\sin\psi - \varepsilon\frac{\pi u}{4K[R + \sqrt{\varepsilon}a(t)]^2\sqrt{C}\omega}\sin(\psi + 2\omega t) \\ & - \varepsilon^{3/2}\frac{\pi\cos(\psi + \omega t)}{2K\sqrt{C}\omega[R + \sqrt{\varepsilon}a(t)]^2}\int_0^t [R + \sqrt{\varepsilon}a(\lambda)]\cos(\psi(\lambda) + \omega\lambda)\sin\omega_i(t - \lambda)\,\mathrm{d}\lambda\end{aligned} \tag{31b}$$

In contrast to (23a) and (23b) which is a *global* model of the slow dynamics, the set (31a) and (31b) represents a *local* model of the slow dynamics since it is valid only in an $O(\sqrt{\varepsilon})$ neighborhood of the 1:1 resonance manifold. As such it is applicable only for the study of the resonance capture interaction between the linear and nonlinear oscillators, in contrast to the model (31a) and (31b) that models the slow flow dynamics for all phases of the motion.

Applying the method of multiple scales to study the local model, we replace the independent time variable with two 'super-slow' and 'slow' variables (since we are dealing with the local model of the slow flow) defined as, $T_1 = \sqrt{\varepsilon}t$ and $T_0 = t$, respectively. We note that the following local analysis represents a further (super-slow) asymptotic analysis within the slow flow dynamics defined by the modulation equations (31a) and (31b). The dependent variables $a(t)$ and $\psi(t)$ are expressed in the following series forms in terms of the slow and super-slow time-scales:

$$\begin{aligned} a &= a_0(T_0, T_1) + \sqrt{\varepsilon}a_1(T_0, T_1) + \varepsilon a_2(T_0, T_1) + O(\varepsilon^{3/2}) \\ \psi &= \psi_0(T_0, T_1) + \sqrt{\varepsilon}\psi_1(T_0, T_1) + \varepsilon\psi_2(T_0, T_1) + O(\varepsilon^{3/2}) \end{aligned} \tag{32}$$

Substituting into (31a) and (31b), and balancing the coefficients of the same order of ε we obtain an hierarchy of subproblems at increasing orders of approximation. In what follows we examine leading order subproblems.

4.1. O(1) approximations

The subproblems of zero-order approximations can be trivially solved:

$$\begin{aligned}\frac{\partial a_0}{\partial T_0} &= 0 \Rightarrow a_0(T_0, T_1) = a_0(T_1) \\ \frac{\partial \psi_0}{\partial T_0} &= 0 \Rightarrow \psi_0(T_0, T_1) = \psi_0(T_1)\end{aligned} \tag{33}$$

where the functions a_0 and ψ_0 are computed by solving the next order of approximation.

4.2. $O(\varepsilon^{1/2})$ approximations

The subproblems governing the first-order approximations are given by:

$$\begin{aligned}\frac{\partial a_1}{\partial T_0} + \frac{\partial a_0}{\partial T_1} &= \frac{\pi}{4K\sqrt{C}} \sin 2(\psi_0 + \omega T_0) - \frac{\pi^2 \beta R}{8K^2} + \frac{\pi^2 \beta R}{8K^2} \cos 2(\psi_0 + \omega T_0) \\ &\quad - \frac{\pi u}{4KR\sqrt{C}\omega} \cos\psi_0 + \frac{\pi u}{4KR\sqrt{C}\omega} \sin(\psi_0 + 2\omega T_0) \frac{\partial \psi_1}{\partial T_0} + \frac{\partial \psi_0}{\partial T_1} = \frac{\pi a_0 \sqrt{C}}{2K}\end{aligned} \tag{34}$$

Taking into account (33), and eliminating secular terms (i.e., terms on the right-hand side that depend only on the super-slow time-scale T_1), we obtain the following solvability relations that govern the unknown functions of the previous order of approximation:

$$\begin{aligned}\frac{\partial a_0}{\partial T_1} &= -\frac{\pi^2 \beta R}{8K^2} - \frac{\pi u}{4KR\sqrt{C}\omega} \cos\psi_0 \\ \frac{\partial \psi_0}{\partial T_1} &= \frac{\pi a_0 \sqrt{C}}{2K}\end{aligned} \tag{35}$$

Differentiating the second of the above relations with respect to T_1 and combining the two equations into a single second-order one, we obtain an equation in the form of a forced pendulum with constant torque,

$$\frac{\partial^2 \psi_0}{\partial T_1^2} + M \cos\psi_0 = -N \tag{36}$$

where,

$$M = \frac{\pi^2 u}{8K^2 R\omega} \quad \text{and} \quad N = \frac{\pi^3 \sqrt{C} \beta R}{16K^3}$$

Eq. (36) provides the leading order analytical approximation to the super-slow flow in the resonance capture regime, since it is governed by the super-slow independent time variable T_1. Depending on the relative values of M and N, the phase portrait of (36) possess (if $M > N$), or does not possess (if $M \leqslant N$) a closed homoclinic loop containing closed super-slow periodic orbits surrounding a stable equilibrium point. This closed loop represents the leading order approximation to the resonance capture domain, e.g., to the set of initial conditions that lead to resonance capture.

When perturbed by higher order terms this domain becomes the attracting region responsible for sustained resonance capture in the dynamical system: for certain initial conditions, trajectories of the system in an $O(\sqrt{\varepsilon})$ neighborhood of the 1:1 resonance manifold get attracted to the region of the loop where they perform multiple oscillations around the attractor. Under other initial conditions trajectories lie outside the homoclinic loop and get repelled away from the attracting region; in this case no resonance capture occurs.

Note that sustained resonance capture is only possible if $M > N$, which leads to the following approximation to lower value for initial velocity of linear oscillator required for sustained resonance capture:

$$u > \frac{2K\beta\omega^3}{\pi\sqrt{C}}$$

This relation indicates that for resonance capture to occur the initial velocity of the directly excited linear oscillator must be above a certain threshold, or equivalently, that the input energy must be above a certain critical threshold. This is in accordance with the conclusions of previous sections (but see also Vakakis and Gendelman, 2001; Quinn, 2002; Vakakis et al., 2003).

After eliminating the secular terms in (34) by means of (35), the $O(\sqrt{\varepsilon})$ approximations to the super-slow flow are computed as follows,

$$\begin{aligned}\alpha_1(T_0,T_1) &= -\frac{\pi}{8K\sqrt{C}\omega}\cos[2(\psi_0+\omega T_0)] + \frac{\pi^2\beta R}{16K^2\omega}\sin[2(\psi_0+\omega T_0)] \\ &\quad - \frac{\pi u}{8KR\sqrt{C}\omega^2}\cos(\psi_0+2\omega T_0) + \widehat{\alpha}(T_1) \\ \psi_1 &= \psi_1(T_1)\end{aligned} \tag{37}$$

where $\widehat{\alpha}(T_1)$ and $\psi_1(T_1)$ denote (yet undetermined) functions of T_1 that result as constants of integration. The new undetermined functions in (37) are computed by eliminating secular terms at the next order of approximation.

4.3. $O(\varepsilon)$ approximations

The subproblems governing the second-order approximations are given by:

$$\begin{aligned}\frac{\partial\alpha_2}{\partial T_0} + \frac{\partial\alpha_1}{\partial T_1} &= \frac{\pi\psi_1}{2K\sqrt{C}}\cos[2(\psi_0+\omega T_0)] - \frac{\pi^2\beta\alpha_0}{8K^2} + \frac{\pi^2\beta\alpha_0}{8K^2}\cos[2(\psi_0+\omega T_0)] - \frac{\pi^2\beta R\psi_1}{4K^2}\sin[2(\psi_0+\omega T_0)] \\ &\quad + \frac{\pi u\alpha_0}{4KR^2\sqrt{C}\omega}\cos\psi_0 + \frac{\pi u\psi_1}{4KR\sqrt{C}\omega}\sin\psi_0 + \frac{\pi u\psi_1}{4KR\sqrt{C}\omega}\cos(\psi_0+2\omega T_0) \\ &\quad - \frac{\pi u\alpha_0}{4KR^2\sqrt{C}\omega}\sin(\psi_0+2\omega T_0) \\ \frac{\partial\psi_2}{\partial T_0} + \frac{\partial\psi_1}{\partial T_1} &= \frac{\pi}{4KR\sqrt{C}} + \frac{\pi}{4KR\sqrt{C}}\cos[2(\psi_0+\omega T_0)] - \frac{\pi^2\beta}{8K^2}\sin[2(\psi_0+\omega T_0)] \\ &\quad + \frac{\pi u}{4KR^2\sqrt{C}\omega}\sin\psi_0 - \frac{\pi u}{4KR^2\sqrt{C}\omega}\sin(\psi_0+2\omega T_0) + \frac{\pi\alpha_1\sqrt{C}}{2K}\end{aligned} \tag{38}$$

Substituting the first of relations (37) into (38), we obtain the following relations:

$$\begin{aligned}\frac{\partial\alpha_2}{\partial T_0} + \frac{\partial\widehat{\alpha_1}}{\partial T_1} &= -\left\{\frac{\pi}{4K\sqrt{C}\omega}\sin[2(\psi_0+\omega T_0)] + \frac{\pi^2\beta R}{8K^2\omega}\cos[2(\psi_0+\omega T_0)] + \frac{\pi u}{4KR\sqrt{C}\omega^2}\sin(\psi_0+2\omega T_0)\right\}\frac{\partial\psi_0}{\partial T_1} \\ &\quad + \frac{\pi\psi_1}{2K\sqrt{C}}\cos[2(\psi_0+\omega T_0)] - \frac{\pi^2\beta\alpha_0}{8K^2} + \frac{\pi^2\beta\alpha_0}{8K^2}\cos[2(\psi_0+\omega T_0)] - \frac{\pi^2\beta R\psi_1}{4K^2}\sin[2(\psi_0+\omega T_0)] \\ &\quad + \frac{\pi u\alpha_0}{4KR^2\sqrt{C}\omega}\cos\psi_0 + \frac{\pi u\psi_1}{4KR\sqrt{C}\omega}\sin\psi_0 + \frac{\pi u\psi_1}{4KR\sqrt{C}\omega}\cos(\psi_0+2\omega T_0) \\ &\quad - \frac{\pi u\alpha_0}{4KR^2\sqrt{C}\omega}\sin(\psi_0+2\omega T_0)\end{aligned} \tag{39a}$$

$$\frac{\partial\psi_2}{\partial T_0}+\frac{\partial\psi_1}{\partial T_1}=\frac{\pi}{4KR\sqrt{C}}+\frac{\pi}{4KR\sqrt{C}}\cos[2(\psi_0+\omega T_0)]-\frac{\pi^2\beta}{8K^2}\sin[2(\psi_0+\omega T_0)]+\frac{\pi u}{4KR^2\sqrt{C}\omega}\sin\psi_0$$
$$-\frac{\pi u}{4KR^2\sqrt{C}\omega}\sin(\psi_0+2\omega T_0)+\frac{\pi\sqrt{C}}{2K}\left\{-\frac{\pi}{8K\sqrt{C}\omega}\cos[2(\psi_0+\omega T_0)]\right.$$
$$\left.+\frac{\pi^2\beta R}{16K^2\omega}\sin[2(\psi_0+\omega T_0)]-\frac{\pi u}{4KR\sqrt{C}\omega^2}\cos(\psi_0+2\omega T_0)\right\}+\frac{\pi\sqrt{C}\widehat{\alpha_1}}{2K} \tag{39b}$$

Eliminating secular terms on the right-hand-side that depend only on the super-slow variable T_1, we obtain the following equations:

$$\begin{aligned}\frac{\partial\widehat{\alpha_1}}{\partial T_1}&=-\frac{\pi^2\beta\alpha_0}{8K^2}+\frac{\pi u\alpha_0}{4KR^2\sqrt{C}\omega}\cos\psi_0+\frac{\pi u\psi_1}{4KR\sqrt{C}\omega}\sin\psi_0\\ \frac{\partial\psi_1}{\partial T_1}&=\frac{\pi}{4KR\sqrt{C}}+\frac{\pi u}{4KR^2\sqrt{C}\omega}\sin\psi_0+\frac{\pi\sqrt{C}\widehat{\alpha_1}}{2K}\end{aligned} \tag{40}$$

Differentiating the second of the above relations with respect to T_1, and combining the two equations into a single second-order one we obtain a nonhomogeneous linear equation with a parameter-dependent coefficient:

$$\frac{\partial^2\psi_1}{\partial T_1^2}-\frac{\pi^2 u}{8K^2R\omega}\psi_1\sin\psi_0=-\frac{\pi^3\sqrt{C}\beta\alpha_0(T_1)}{16K^3} \tag{41}$$

Introducing the scaled independent super-slow variable $\tau=\sqrt{h_1}T_1$, and defining the quantities,

$$h_1=\frac{\pi^2 u}{8K^2R\omega},\quad h_2=\frac{\pi^3\sqrt{C}\beta R}{16K^3},\quad \text{and}\quad a_0(T_1)=\tilde{a}_0(\tau)=\frac{2K\sqrt{h_1}}{\pi\sqrt{C}}\frac{\partial\psi_0}{\partial\tau},$$

Eq. (41) becomes,

$$\frac{\partial^2\psi_1}{\partial\tau^2}-\psi_1\sin\psi_0=-\frac{h_2\tilde{\alpha}_0(\tau)}{h_1R} \tag{42}$$

where $\psi_i=\psi_i(\tau)$. This equation provides a correction to the leading order approximation (36) of the super-slow flow. It turns out that due to its linear structure this equation can be solved analytically.

One homogeneous solution of (42) is given by:

$$\psi_1^{(1h)}=-\frac{\partial\psi_0}{\partial\tau} \tag{43a}$$

A second linearly independent homogeneous solution can be obtained by considering the equation for the Wroskian of (42), leading to the following analytical expression:

$$\psi_1^{(2h)}=\psi_1^{(1h)}(\tau)\int^{\tau}\mathrm{d}z\,\psi_1^{(1h)}(z)]^{-2} \tag{43b}$$

These two linearly independent homogeneous solutions are used to compute a particular integral by the method of variation of parameters which completes the solution:

$$\psi_1(\tau)=\left[\gamma-\frac{h_2 2K}{\sqrt{h_1}R\pi\sqrt{C}}\int^{\tau}\psi_1^{(1h)}(z)\psi_1^{(2h)}(z)\,\mathrm{d}z\right]\psi_1^{(1h)}(\tau)+\left[\delta+\frac{h_2 2K}{\sqrt{h_1}R\pi\sqrt{C}}\int^{\tau}\psi_1^{(1h)}(z)\psi_1^{(1h)}(z)\,\mathrm{d}z\right]\psi_1^{(2h)}(\tau) \tag{44}$$

The above relation provides a general solution to the $O(\sqrt{\varepsilon})$ super-slow flow approximation (42), provided that a solution of (36) has been computed. The constant coefficients γ and δ in (44) are selected so that asymptotic properties of the overall solution as $\tau \to \pm\infty$ are satisfied.

As an application, we will use the general formula (44) to compute the $O(\sqrt{\varepsilon})$ perturbation of the homoclinic loop of the forced pendulum (36), assuming that $u > 2K\beta\omega^3/(\pi\sqrt{C})$ (so that the homoclinic loop and the resonance capture region exist). A similar procedure can be followed to compute the perturbations of the periodic orbits inside and the nonperiodic orbits outside the unperturbed homoclinic loop of (36). The methodology is similar to that used by Vakakis (1994) to analyze the splitting of the manifolds of a harmonically forced pendulum.

The unperturbed phase plane $(\psi_0, \frac{\partial\psi_0}{\partial\tau})$ of (36) contains unstable and stable fixed points, and a homoclinic loop that connects the unstable fixed point with itself, and encircles a family of periodic orbits that surround the stable fixed point. When $O(\sqrt{\varepsilon})$ terms are taken into account, we expect the degenerate homoclinic loop to breakdown and to be decomposed into one-dimensional stable and unstable invariant manifolds of the hyperbolic unstable fixed point; we denote the solutions on the invariant manifolds by $\psi^{(\mathrm{s,u})}(\tau)$, where the superscripts (s) and (u) indicate solutions on the stable and unstable manifolds, respectively We denote the zeroth order approximations to these special solutions by $\psi_0^{(\mathrm{s,u})}(\tau)$, with the initial conditions specified at $\tau = 0$. We compute the zeroth order approximations by solving Eq. (36); although an exact analytic solution can be derived in terms of exponential functions (by integrating the equation by quadratures), in this work we resorted to direct numerical solutions of (36) to approximate them. The $O(1)$ solutions on the homoclinic loop satisfy the limiting conditions:

$$\lim_{\tau\to\infty} \psi_0^{(\mathrm{s})}(\tau) = \cos^{-1}(-N/M) \quad \text{and} \quad \lim_{\tau\to-\infty}\psi_0^{(u)}(\tau) = \cos^{-1}(-N/M) \tag{45}$$

The $O(\sqrt{\varepsilon})$ perturbations $\psi_1^{(\mathrm{s,u})}(\tau)$ of $\psi_0^{(\mathrm{s,u})}(\tau)$ are computed by the general expression (44) with appropriate evaluation of the constants γ and δ. We express these perturbations as,

$$\begin{aligned}\psi_1^{(\mathrm{s,u})}(\tau) = {}& \left[\gamma^{(\mathrm{s,u})} - \frac{h_2 2K}{\sqrt{h_1}R\pi\sqrt{C}}\int_0^\tau \psi_1^{(\mathrm{s,u})1h}(z)\psi_1^{(\mathrm{s,u})2h}(z)\,\mathrm{d}z\right]\psi_1^{(\mathrm{s,u})1h}(\tau) \\ & + \left[\delta^{(\mathrm{s,u})} + \frac{h_2 2K}{\sqrt{h_1}R\pi\sqrt{C}}\int_0^\tau \psi_1^{(\mathrm{s,u})1h}(z)\psi_1^{(\mathrm{s,u})1h}(z)\,\mathrm{d}z\right]\psi_1^{(\mathrm{s,u})2h}(\tau)\end{aligned} \tag{46}$$

where,

$$\psi_1^{(\mathrm{s,u})1h} = -\frac{\partial\psi_0^{(\mathrm{s,u})}}{\partial\tau}, \quad \psi_1^{(\mathrm{s,u})2h} = \psi_1^{(\mathrm{s,u})1h}(\tau)\int^\tau \mathrm{d}z[\psi_1^{(\mathrm{s,u})1h}(z)]^{-2} \tag{47}$$

are the corresponding linearly independent homogeneous solutions. The first term in (46) is well behaved since it reaches a bounded motion as $\tau \to \pm\infty$. Considering the second term, however, as $\tau \to \pm\infty$ the function $\psi_1^{(\mathrm{s,u})2h}(\tau)$ diverges, whereas the definite integral reaches a finite limit; it can be shown that in order to obtain bounded limits for $\psi_1^{(\mathrm{s,u})}(\tau)$ as $\tau \to \pm\infty$, it is necessary to compute the constants $\delta^{(\mathrm{s,u})}$ as follows:

$$\begin{aligned}\delta_\mathrm{s} &= -\frac{h_2 2K}{\sqrt{h_1}R\pi\sqrt{C}}\int_0^{+\infty} \psi_1^{(\mathrm{s})1h}(z)\psi_1^{(\mathrm{s})1h}(z)\,\mathrm{d}z \\ \delta_\mathrm{u} &= -\frac{h_2 2K}{\sqrt{h_1}R\pi\sqrt{C}}\int_0^{-\infty} \psi_1^{(\mathrm{u})1h}(z)\psi_1^{(\mathrm{u})1h}(z)\,\mathrm{d}z\end{aligned} \tag{48}$$

To evaluate the constants $\gamma^{(\mathrm{s,u})}$ we observe that due to the first of relations (47), motions on the stable and unstable manifolds can be approximately expressed in the form:

$$\begin{aligned}\psi^{(s,u)}(\tau;\varepsilon) &= \psi_0^{(s,u)}(\tau) - \sqrt{\varepsilon}\gamma^{(s,u)}\frac{\partial\psi_0^{(s,u)}(\tau)}{\partial\tau} + \sqrt{\varepsilon}H^{(s,u)}(\tau) + O(\varepsilon) \\ &= \psi_0^{(s,u)}(\tau - \sqrt{\varepsilon}\gamma^{(s,u)} + O(\varepsilon)) + \sqrt{\varepsilon}H^{(s,u)}(\tau) + O(\varepsilon)\end{aligned} \tag{49}$$

where $H^{(s,u)}(\tau)$ denote the $O(\sqrt{\varepsilon})$ terms that do not depend on $\gamma^{(s,u)}$. From the above expression we note that if $\gamma^{(s,u)} \neq 0$ unwanted (and unjustified) time shifts are introduced in the $O(1)$ approximations. Since the outlined perturbation analysis is carried out specifying the initial conditions at $\tau = 0$ the time-shift in (49) must be eliminated by setting:

$$\gamma^{(s)} = \gamma^{(u)} = 0 \tag{50}$$

Combining the previous results, correct to $O(\sqrt{\varepsilon})$, we can write the following expressions for the solutions on the perturbed stable and unstable invariant manifolds:

$$\begin{aligned}\psi^{(s,u)}(\tau) = \psi_0^{(s,u)}(\tau) + \sqrt{\varepsilon}\Bigg\{&\left[-\frac{h_2 2K}{\sqrt{h_1}R\pi\sqrt{C}}\int_0^{\tau}\psi_1^{(s,u)1h}(z)\psi_1^{(s,u)2h}(z)\,\mathrm{d}z\right]\psi_1^{(s,u)1h}(\tau) \\ &- \sqrt{\varepsilon}\left[\frac{h_2 2K}{\sqrt{h_1}R\pi\sqrt{C}}\int_{\tau}^{\pm\infty}\psi_1^{(s,u)1h}(z)\psi_1^{(s,u)1h}(z)\,\mathrm{d}z\right]\psi_1^{(s,u)2h}(\tau)\Bigg\} + O(\varepsilon)\end{aligned} \tag{51}$$

In Fig. 6 we depict the plot of (51) in the projection (ψ', ψ) of the phase space of the slow flow (where primes denote differentiation with respect to τ). Comparing this plot (solid line) with the unperturbed homoclinic loop (dashed line) we see that the $O(\sqrt{\varepsilon})$ perturbations indeed breakup the degenerate homoclinic structure and define the path through which trajectories are 'trapped' in the resonance capture region. We note that since the analysis includes only terms up to $O(\sqrt{\varepsilon})$, the results are expected to be accurate only up to times of $O(1/\sqrt{\varepsilon})$, and to diverge after that. By carrying the perturbation analysis to higher order one should be able to extend the validity of the asymptotic results to longer times, and study the resonance capture region more accurately (including the 'trapping' and 'escape' paths leading into and out of the region in phase space of sustained capture).

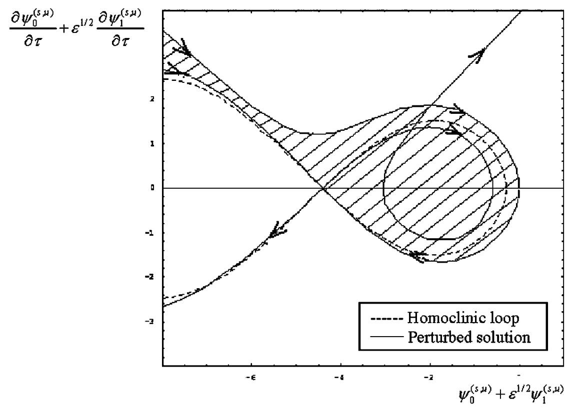

Fig. 6. Projection of the phase space of the super-slow flow incorporating $O(\sqrt{\varepsilon})$ corrections; shadowed region indicates the approximate region of resonance capture and the 'trapping' path for the trajectories.

5. Energy pumping resulting from multiple, simultaneous resonant interactions

In this section we will present an alternative way of nonlinear resonance interactions leading to energy pumping, realized in multi-degree-of-freedom (MDOF) essentially nonlinear end attachments. As shown below, this type of MDOF nonlinear end attachments leads to simultaneous resonance interactions of multiple nonlinear normal modes (NNMs) with multiple normal modes of the main linear system; such simultaneous resonance interactions can lead to multi-mode energy pumping from the linear system to the attachment through a mechanism different than resonance capture cascading.

To demonstrate this alternative way of resonance interactions we consider the system of Fig. 7, consisting of a two-DOF linear system weakly coupled to a three-DOF attachment. We assume that there exists weak viscous damping, and that the ungrounded end attachment possesses essentially nonlinear (nonlinearizable) stiffnesses. In the limit of zero small parameter, $\varepsilon \to 0$, this system decouples into two subsystems: (a) a linear one, possessing two (in-phase and an anti-phase) normal modes; and (b) an essentially nonlinear one, possessing a rigid body mode corresponding to $\omega = 0$, and two nonlinear normal modes (NNMs) (Vakakis et al., 1996). By NNMs we denote synchronous free periodic motions of the nonlinear system, analogous to the normal modes of classical linear vibration theory. In Fig. 8 we depict typical *backbone curves* of the two NNMs of the decoupled nonlinear system (corresponding to $\varepsilon = 0$), e.g., the frequency–energy relationships of the free periodic motions. Considering the positions of the natural frequencies of the decoupled linear system, we conclude that depending on the energy of the vibration there exists the possibility for multiple internal resonances between the NNMs and the linear normal modes (points A, B, and C in Fig. 8). Each of these internal resonances is capable of producing 1:1 resonance capture phenomena of the type considered in earlier sections. Moreover, the possibility exists for the occurrence of simultaneous resonance captures between each

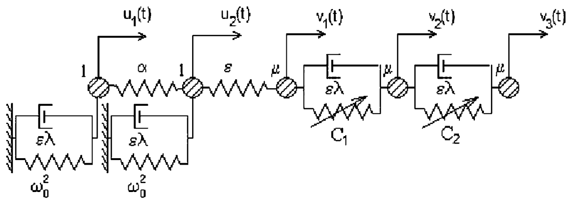

Fig. 7. Linear system with MDOF essentially nonlinear end attachment.

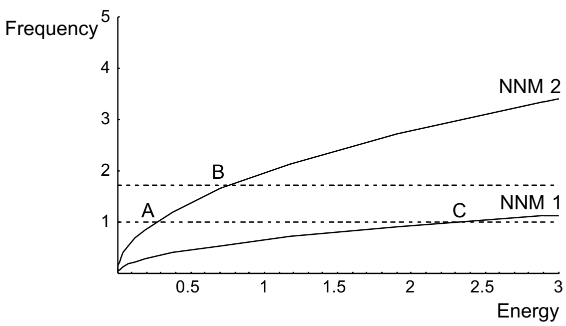

Fig. 8. Backbone curves of the NNMs of the decoupled system corresponding to $\varepsilon = 0$, and possibilities for multiple resonance captures.

NNM and one of the normal modes of the linear system, which leads to an alternative way to resonance capture cascading for multi-mode nonlinear energy pumping from the linear system to the nonlinear attachment.

To demonstrate energy pumping in the system of Fig. 7, we express the equations of motion in terms of the modal coordinates of the linear subsystem:

$$\begin{aligned}
&\ddot{x}_1 + \varepsilon\lambda\dot{x}_1 + \left(\omega_1^2 + \frac{\varepsilon}{2}\right)x_1 - \varepsilon\left(\frac{x_2}{2} + v_1\right) = 0 \\
&\ddot{x}_2 + \varepsilon\lambda\dot{x}_2 + \left(\omega_2^2 + \frac{\varepsilon}{2}\right)x_2 - \varepsilon\left(\frac{x_2}{2} - v_1\right) = 0 \\
&\mu\ddot{v}_1 + \varepsilon\lambda(\dot{v}_1 - \dot{v}_2) + \varepsilon\left(v_1 + \frac{x_2 - x_1}{2}\right) + C_1(v_1 - v_2)^3 = 0 \\
&\mu\ddot{v}_2 + \varepsilon\lambda(2\dot{v}_2 - \dot{v}_1 - \dot{v}_3) + C_1(v_2 - v_1)^3 + C_2(v_2 - v_3)^3 = 0 \\
&\mu\ddot{v}_3 + \varepsilon\lambda(\dot{v}_3 - \dot{v}_2) + C_2(v_3 - v_2)^3 = 0
\end{aligned} \tag{52}$$

In Fig. 9 we depict the transient responses and their Fast Fourier Transforms (FFTs) of the system with parameters,

$$\varepsilon = 0.2, \quad \mu = 0.2, \quad C_1 = C_2 = 0.3, \quad \lambda = 0.25, \quad \omega_1^2 = 1, \quad \omega_2^2 = 3$$

and initial conditions $\dot{x}_1(0) = \dot{x}_2(0) = 5.0$, with all other displacements and velocities zero. This set of initial conditions corresponds to impulsive excitations of both modes of the linear subsystem. In the plots of Fig. 9 we depict the linear modal coordinates $x_1(t)$ and $x_2(t)$, as well as the nonlinear modal coordinates $z_1(t)$ and $z_2(t)$, defined as,

$$z_1(t) = [v_2(t) - v_3(t)] - [v_1(t) - v_2(t)], \quad z_2(t) = [v_2(t) - v_3(t)] + [v_1(t) - v_2(t)]$$

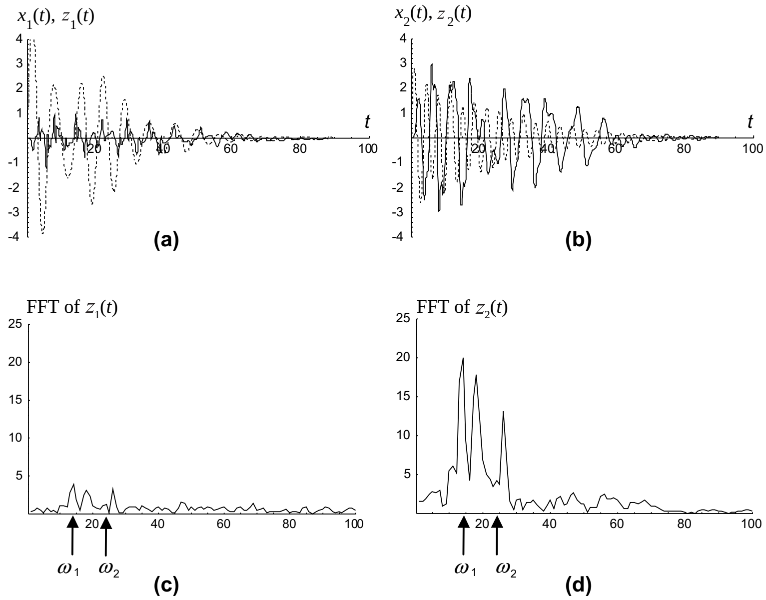

Fig. 9. Transient dynamics in terms of modal coordinates: (a) $x_1(t)$ (- - -) and $z_1(t)$ (–); (b) $x_2(t)$ (- - -) and $z_2(t)$ (–); (c) FFT of $z_1(t)$; (d) FFT of $z_2(t)$.

We note as this point that for the parameters chosen the decoupled nonlinear system possesses the two *similar NNMs* (corresponding to straight lines in the configuration space of the system, Vakakis et al., 1996) $[v_2(t) - v_3(t)] = [v_1(t) - v_2(t)]$ and $[v_2(t) - v_3(t)] = -[v_1(t) - v_2(t)]$; hence, $z_1(t)$ denotes the response of the in-phase similar NNM, whereas $z_2(t)$ the response of the out-of-phase similar NNM.

Studying the transient responses we note that although there is no detectable resonance capture cascading (e.g., sequential resonance captures with the linear modes of the type studied in previous sections), the transient responses $z_1(t)$ and $z_2(t)$ possess at least three strong frequency components, two of which are close to the two natural frequencies of the linear subsystem, and one lying in between. Hence, the NNMs of the MDOF attachment seem to engage in resonance capture with both modes of the linear system, without, however, engaging in resonance capture cascading. The additional mode appearing in the FFTs of $z_1(t)$ and $z_2(t)$ is generated due to the NNM bifurcations that occur due to the internal resonances between the linear and nonlinear subsystems (points A, B, and C in Fig. 8, Vakakis et al., 2003). As a result, energy pumping from both linear modes to the nonlinear system takes place, without the occurrence of resonance capture cascading.

6. Concluding remarks

The $O(\sqrt{\varepsilon})$ corrections considered in the previous local asymptotic analysis do not capture radiation effects from the nonlinear attachment to the chain; by including $O(\varepsilon)$ terms in the asymptotic analysis one should be able to take into account the integral terms in (31a) and (31b) that model the energy radiation effects, and, hence, one should be able to also study the 'drift' of the stable equilibrium point in the resonance capture region as the energy of the attachment diminishes due to damping dissipation and radiation to the chain. Indeed, as time increases one anticipates the entire resonance capture region to gradually shrink to zero as the energy of the nonlinear attachment asymptotically tends to zero.

In addition, by carrying the analysis to higher orders of approximation (a tedious exercise that, however, could be conveniently performed through the use of computer algebra), one could be able to more accurately determine the domain of attraction of the resonance capture region. This would establish the initial conditions of the system for which passive energy pumping can occur, which would be helpful from a design point of view.

We note that the local asymptotic results presented in this paper can be applied to study resonance capture interactions of essentially nonlinear attachments with a more general class of finite-DOF linear systems; this can be achieved by reducing the problem to a series of resonance captures of the attachment with individual structural modes. Hence, the presented analysis is general, and of wide applicability.

Finally, by considering the interaction of a linear system with a MDOF nonlinear end attachment we presented an alternative way to resonance capture cascading for multi-mode nonlinear energy pumping; this is based on simultaneous resonant interactions of multiple NNMs of the end attachment with multiple normal modes of the linear system. This new design paves the way for more effective multi-mode energy pumping from the linear system to the attachment.

Acknowledgements

This work was supported in part by the grant for basic research 'Thales' awarded by the National Technical University of Athens.

Appendix A. Analytic approximation of the motion of the nonlinear attachment in the limit of small oscillations

Considering the response of the nonlinear attachment after the energy pumping phenomenon has taken place, the nonlinear oscillator performs slowly decaying oscillations due to damping dissipation with small amplitude. Because of the small amplitude of the attachment we neglect the cubic term in the last equation of system (1), and consider the following approximate system:

$$\begin{aligned}
&\ddot{u}_1 + u_1(\omega_0^2 + 2\alpha) - \alpha u_2 = 0\\
&\ddot{u}_n + u_n(\omega_0^2 + 2\alpha) - \alpha(u_{n+1} - u_{n-1}) = 0, \qquad n = 2, 3, \ldots, N-1\\
&\ddot{u}_N + u_N(\omega_0^2 + \alpha + \varepsilon) - \alpha u_{N-1} - \varepsilon v = 0\\
&\ddot{v} + \varepsilon\beta\dot{v} + \varepsilon(v - u_N) = 0
\end{aligned} \tag{A.1}$$

We write this system into the following matrix:

$$I\ddot{x} + Kx + B\dot{x} = 0, \quad x(t) \in \Re^{N+1} \tag{A.2}$$

where K and B are $(N+1)\times(N+1)$ stiffness and damping matrices. We want to find the eigenvalues and eigenvectors of the symmetric matrix K. This matrix can be split up into the sum of two symmetric matrices:

$$K = K_0 + \varepsilon K_1 \tag{A.3}$$

where,

$$K_0 = \begin{pmatrix} \omega_0^2 + 2\alpha & -\alpha & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ -\alpha & \omega_0^2 + 2\alpha & -\alpha & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & -\alpha & \omega_0^2 + \alpha & & & 0 \\ 0 & \cdots & \cdots & \cdots & & 0 & 0 & & & 0 \end{pmatrix} \text{ and}$$

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

We denote the eigenvalues of K_0 by,

$$\lambda_i^{(0)} = \omega_i^2 > 0, \quad 1 \leqslant i \leqslant N, \quad \lambda_{N+1}^{(0)} = 0 \tag{A.4}$$

and the corresponding eigenvectors by:

$$\phi_i^{(0)} = \begin{pmatrix} \phi_{i,1}^{(0)} \\ \phi_{i,2}^{(0)} \\ \vdots \\ \phi_{i,N}^{(0)} \\ 0 \end{pmatrix}, \quad \phi_{i,k}^{(0)} \neq 0, \quad 1 \leqslant i \leqslant N, \quad \phi_{N+1}^{(0)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{A.5}$$

We now consider the eigenvalue problem $K\varphi_i = \lambda_i\varphi_i$, and approximate the eigensolutions in the following perturbation form,

$$\begin{aligned}
\lambda_i &= \lambda_i^{(0)} + \varepsilon\lambda_i^{(1)} + O(\varepsilon^2)\\
\varphi_i &= \varphi_i^{(0)} + \varepsilon\varphi_i^{(1)} + O(\varepsilon^2)
\end{aligned} \tag{A.6}$$

where,

$$\begin{aligned}\lambda_i^{(1)} &= \varphi_i^{(0)\mathrm{T}} K_1 \varphi_i^{(0)} = (\varphi_{i,N}^{(0)})^2, \quad 1 \leqslant i \leqslant N \\ \lambda_{N+1}^{(1)} &= \varphi_{N+1}^{(0)\mathrm{T}} K_1 \varphi_{N+1}^{(0)} = 1\end{aligned} \tag{A.7a}$$

and,

$$\begin{aligned}\varphi_i^{(1)} &= \sum_{m\neq i}^{N} \frac{\varphi_m^{(0)\mathrm{T}} K_1 \varphi_i^{(0)}}{\omega_i^2 - \omega_m^2} \varphi_m^{(0)} + \frac{\varphi_{N+1}^{(0)\mathrm{T}} K_1 \varphi_i^{(0)}}{\omega_i^2} \varphi_{N+1}^{(0)}, \quad 1 \leqslant i \leqslant N \\ \varphi_{N+1}^{(1)} &= \sum_{m=1}^{N} \frac{\varphi_m^{(0)\mathrm{T}} K_1 \varphi_{N+1}^{(0)}}{-\omega_m^2} \varphi_m^{(0)}\end{aligned} \tag{A.7b}$$

Taking into account the definition of matrix K_1 the corrections to the eigenvectors are expressed as:

$$\begin{aligned}\varphi_i^{(1)} &= \sum_{m\neq i}^{N} \frac{\varphi_{m,N}^{(0)} \varphi_{i,N}^{(0)}}{\omega_i^2 - \omega_m^2} \varphi_m^{(0)} - \frac{\varphi_{N+1,N+1}^{(0)} \varphi_{i,N}^{(0)}}{\omega_i^2} \varphi_{N+1}^{(0)}, \quad 1 \leqslant i \leqslant N \\ \varphi_{N+1}^{(1)} &= -\sum_{m=1}^{N} \frac{\varphi_{m,N}^{(0)\mathrm{T}} \varphi_{N+1,N+1}^{(0)}}{-\omega_m^2} \varphi_m^{(0)}\end{aligned} \tag{A.7c}$$

Therefore, through first order the eigenvalues of the perturbed problem are expressed as:

$$\begin{aligned}\lambda_i &= \omega_i^2 + \varepsilon(\varphi_{i,N}^{(0)})^2 + O(\varepsilon^2), \quad 1 \leqslant i \leqslant N \\ \lambda_{N+1}^{(1)} &= \varepsilon + O(\varepsilon^2)\end{aligned} \tag{A.8}$$

In order to diagonalize the stiffness matrix K, we consider the $(N+1) \times (N+1)$ matrix Q composed of the $(N+1)$ normalized eigenvectors. Because K is symmetrical any two different normalized eigenvectors are orthogonal vectors satisfying the relations,

$$Q^\mathrm{T} Q = I, \quad Q^\mathrm{T} K Q = D$$

where D is the diagonal matrix of eigenvalues. Introducing the coordinate transformation,

$$q = Q^\mathrm{T} x \tag{A.9}$$

and pre-multiplying (A.2) by Q^T we express the system into the following simplified form:

$$\ddot{q} + Dq + Q^\mathrm{T} B Q \dot{q} = 0 \tag{A.10}$$

where,

$$Q^\mathrm{T} B Q = \varepsilon\beta \begin{pmatrix} \phi_{1,N+1}^2 & \phi_{1,N+1}\phi_{2,N+1} & \cdots & \phi_{1,N+1}\phi_{N+1,N+1} \\ \phi_{1,N+1}\phi_{2,N+1} & \phi_{2,N+1}^2 & \cdots & \phi_{2,N+1}\phi_{N+1,N+1} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{1,N+1}\phi_{N+1,N+1} & \phi_{2,N+1}\phi_{N+1,N+1} & \cdots & \phi_{N+1,N+1}^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \varepsilon\beta \end{pmatrix} + O(\varepsilon^2)$$

Therefore, to the first approximation the set of equations (A.10) can be expressed as:

$$\begin{aligned}&\ddot{q}_i + \lambda_i q_i = 0, \quad 1 \leqslant i \leqslant N \\ &\ddot{q}_{N+1} + \lambda_{N+1} q_{n+1} + \varepsilon\beta \dot{q}_{N+1} = 0\end{aligned} \tag{A.11}$$

Considering the form of the above set of uncoupled equations, we conclude that the approximate frequencies of the system (A.1) for sufficiently small amplitudes (e.g., after the energy pumping regime) are given by:

$$\sqrt{\lambda_i} = \sqrt{\omega_i^2 + \varepsilon(\phi_{i,N}^{(0)})^2}, \quad 1 \leqslant i \leqslant N$$
$$\omega_{N+1} = \sqrt{\varepsilon}\sqrt{1 - \frac{\varepsilon\beta^2}{4}} \tag{A.12}$$

This result indicates that after the resonance capture regime the nonlinear attachment oscillates with small amplitude and approximate frequency of oscillation equal to $\sqrt{\varepsilon(1 - \varepsilon\beta^2/4)}$ and damping ratio equal to $\zeta = \beta\varepsilon/2$.

References

Arnold, V.I. (Ed.), 1988. Dynamical systems III. In: Encyclopaedia of Mathematical Sciences. Springer Verlag, Berlin and New York.

Barkham, P.G.D., Soudack, A.C., 1969. An extension of the method of Kryloff and Bogoliuboff. International Journal of Control 10, 377–392.

Barkham, P.G.D., Soudack, A.C., 1970. approximate solutions of nonlinear nonautonomous second-order differential equations. International Journal of Control 11, 101–114.

Coppola, V.T., Rand, R.H., 1990. Averaging using elliptic functions: approximation of limit cycles. Acta Mechanica 81, 125–142.

Quinn, D., 1997. Resonance capture in a three degree of freedom mechanical system. Nonlinear Dynamics 14, 309–333.

Quinn, D., 2002. Resonant Dynamics in Strongly Nonlinear Systems. Nonlinear Dynamics, preprint. Submitted for publication.

Vakakis, A.F., 1994. Exponentially small splittings of manifolds in a rapidly forced duffing system. Journal of Sound and Vibration 170 (1), 119–129.

Vakakis, A.F., Gendelman, O., 2001. Energy 'pumping' in nonlinear mechanical oscillators II: resonance capture. Journal of Applied Mechanics 68 (1), 42–48.

Vakakis, A.F., Rand, R.H., 2004. Nonlinear dynamics of a system of coupled oscillators with essential stiffness nonlinearities. International Journal of Nonlinear Mechanics 39, 1079–1091.

Vakakis, A.F., Manevitch, L.I., Mikhlin, Yu.V., Pilipchuk, V.N., Zevin, A.A., 1996. Normal Modes and Localization in Nonlinear Systems. Wiley Interscience, New York.

Vakakis, A.F., Manevitch, L.I., Gendelman, O., Bergman, L., 2003. Dynamics of linear discrete systems connected to local essentially nonlinear attachments. Journal of Sound and Vibration 264, 559–577.